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## Entangled coherent states for systems with $SU(2)$ and $SU(1, 1)$ symmetries

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**Abstract.** Entangled  $SU(2)$  and  $SU(1, 1)$  coherent states are developed as superpositions of multiparticle  $SU(2)$  and  $SU(1, 1)$  coherent states. As a special example of entangled  $SU(2)$  coherent states, entangled binomial states are introduced and these entangled binomial states enable the contraction from entangled  $SU(2)$  coherent states to entangled harmonic oscillator coherent states. Entangled  $SU(2)$  coherent states are discussed in the context of pairs of qubits. We also introduce the entangled negative binomial states and entangled squeezed states as examples of entangled  $SU(1, 1)$  coherent states. A method for generating the entangled  $SU(2)$  and  $SU(1, 1)$  coherent states is discussed and degrees of entanglement calculated. Two types of  $SU(1, 1)$  coherent states are discussed in each case: Perelomov coherent states and Barut–Girardello coherent states.

### 1. Introduction

Entanglement is a key feature of quantum mechanics which decisively distinguishes quantum physics from classical physics. The phenomenon of entanglement is dramatically illustrated in the rapidly developing field of quantum information but is important in other contexts as well. In fact, rapid developments in quantum information theory motivate studies of entanglement, where entanglement itself is a resource for performing quantum computation which exceeds the capability of its classical counterpart; examples include factorization of integers [1] and combinatorics [2]. To begin our discussion of entangled coherent states, we first begin by discussing qubits and entanglements of qubit states, including entanglements of nonorthogonal qubit states.

Qubits are the basic elements of quantum information technology, in the same way that bits are basic units of information in computers. Whereas bits are binary digits, qubits are spin- $\frac{1}{2}$ , or two-level, quantum systems. The advantage of quantum computing over classical computing is the capacity for producing entangled qubits: the large state space available for entangled qubits enables certain problems, thought not to be computable on classical computers, to be solved on quantum computers [3].

A bit can be in an off, or '0', state or an on, or '1', state, but the qubit can be in a superposition of an off, or '|0>', state and an on, or '|1>', state. We can represent such a state

(a general qubit) by

$$\begin{aligned} |\theta, \phi\rangle &= \exp\left[-\frac{\theta}{2}(\sigma_+ e^{-i\phi} - \sigma_- e^{i\phi})\right] |1\rangle \\ &= \cos\frac{\theta}{2}|1\rangle + e^{i\phi} \sin\frac{\theta}{2}|0\rangle \end{aligned} \quad (1)$$

up to a global phase. Here  $\sigma_{\pm} = \sigma_x \pm i\sigma_y$ , for  $\sigma_x$  and  $\sigma_y$  Pauli matrices. In fact such a state is an  $SU(2)$  coherent state, also known as an atomic coherent state [4,5]. Two qubits, prepared in a product state, could then be expressed as  $|\theta_1, \phi_1\rangle \otimes |\theta_2, \phi_2\rangle$ . However, quantum computation is based on the exploitation of entanglement, and such product states are of limited value in quantum information. The simplest extension of this arbitrary two-qubit product state to a two-qubit entangled state is the unnormalized state

$$\cos\theta|\theta_1, \phi_1\rangle \otimes |\theta_2, \phi_2\rangle + e^{i\phi} \sin\theta|\theta'_1, \phi'_1\rangle \otimes |\theta'_2, \phi'_2\rangle \quad (2)$$

which is a product state for  $\theta$  a multiple of  $\pi/2$ . Of course the Bell states [6, 7]

$$\begin{aligned} |\Phi\rangle_{\pm} &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |0\rangle \pm |1\rangle \otimes |1\rangle) \\ |\Psi\rangle_{\pm} &= \frac{1}{\sqrt{2}}(|0\rangle \otimes |1\rangle \pm |1\rangle \otimes |0\rangle) \end{aligned} \quad (3)$$

are special cases of the general state (2).

The  $SU(2)$  coherent states form an overcomplete basis for the Hilbert space, and the qubit states correspond to spin- $\frac{1}{2}$  representations of  $SU(2)$ . There are therefore subtleties concerning this entanglement of non-orthogonal states: such subtleties have been considered with respect to entangled coherent states (or superpositions of multimode coherent states), where the coherent states have been Heisenberg–Weyl, or harmonic oscillator, coherent states [8–25]. Our objective here is to introduce and analyse entangled  $SU(2)$  and  $SU(1, 1)$  coherent states.

The entangled  $SU(1, 1)$  coherent states are closely related to the  $SU(2)$  coherent states because the algebra  $su(1, 1)$  and  $su(2)$  are so similar. However, there are two commonly considered coherent states for  $SU(1, 1)$ . One  $SU(1, 1)$  coherent state is the analogue of the harmonic oscillator coherent state achieved by displacing the vacuum state and the  $SU(2)$  coherent state obtained by ‘rotating’ the lowest- or highest-weight state. The analogous  $SU(1, 1)$  coherent state is obtained via an  $SU(1, 1)$  transformation of lowest-weight state. This  $SU(1, 1)$  coherent state is a member of Perelomov’s category of generalized coherent state, and we refer to this state as a Perelomov  $SU(1, 1)$  coherent state [5]. The second  $SU(1, 1)$  coherent state, introduced by Barut and Girardello [26], is the analogue of the harmonic oscillator coherent state being an eigenstate of the annihilation operator; an  $SU(2)$  coherent state of the Barut–Girardello type does not exist due to the finiteness of the  $SU(2)$  Hilbert space. We treat entangled  $SU(1, 1)$  coherent states of both the Perelomov and Barut–Girardello types; such states can be regarded as deformations of Heisenberg–Weyl coherent states but dual to one another. As a special case of the entangled Perelomov  $SU(1, 1)$  coherent state, we obtain superpositions of squeezed vacuum states [27] and entangled squeezed states [14]. Squeezed states are significant in quantum-limited measurements, quantum communications and the exotic spectroscopy of atoms [28].

In association with the parity operator, this paper first develops parity coherent states and entangled  $SU(2)$  and  $SU(1, 1)$  coherent states in section 2. Section 3 generalizes the parity coherent states and considers nonlinear  $SU(2)$  and  $SU(1, 1)$  coherent states. In section 4, we discuss how to represent entangled  $SU(2)$  and  $SU(1, 1)$  coherent states in Fock space, and we have obtained the entangled binomial states, entangled negative binomial states and entangled squeezed states as well as the contraction of entangled  $SU(2)$  and  $SU(1, 1)$  coherent

states to entangled harmonic oscillator coherent states. Section 5 investigates how to generate the entangled coherent states in Hamiltonian systems. Section 6 discusses the degree of entanglement for these entangled coherent states, and a conclusion is given in section 7. The appendix gives the most general entangled  $SU(2)$  and  $SU(1, 1)$  coherent states with certain special cases, including the quantum Fourier transform state of a product multiparticle  $SU(2)$  coherent state required by Shor's algorithm [1].

## 2. Entangled $SU(2)$ and $SU(1, 1)$ coherent states

### 2.1. Entangled coherent states of the harmonic oscillator

For the harmonic oscillator, a general unnormalized two-particle entangled coherent state can be expressed as [11]

$$\cos \theta |\alpha_1\rangle \otimes |\alpha_2\rangle + e^{i\phi} \sin \theta |\alpha'_1\rangle \otimes |\alpha'_2\rangle = [\cos \theta \exp(\vec{\alpha} \cdot \vec{a}^\dagger - \vec{\alpha}^* \cdot \vec{a}) + e^{i\phi} \sin \theta \exp(\vec{\alpha}' \cdot \vec{a}^\dagger - \vec{\alpha}'^* \cdot \vec{a})] |0, 0\rangle \quad (4)$$

for  $\vec{a} \equiv (a_1, a_2)$ ,  $\vec{\alpha} \equiv (\alpha_1, \alpha_2)$  and  $|0, 0\rangle = |0\rangle_1 \otimes |0\rangle_2$  the vacuum state. It is particularly helpful to concentrate on the balanced entangled coherent state

$$2^{-1/2} [|\alpha_1\rangle \otimes |\alpha_2\rangle + e^{i\phi} |-\alpha_1\rangle \otimes |-\alpha_2\rangle] \quad (5)$$

where the word 'balanced' refers to each component in the superposition having coefficients of the same magnitude. For  $\phi = \pi/2$  such a state can, in principle, be generated via a nonlinear interferometer [9, 22] and is an eigenstate of the canonically transformed pair annihilation operator [16]

$$\vec{\tilde{a}} \equiv \Pi \vec{a} \quad (6)$$

with

$$\Pi = e^{i\pi \vec{a}^\dagger \cdot \vec{a}} \quad \Pi \vec{a} \Pi = -\vec{a}. \quad (7)$$

Thus, the entangled coherent state (5) is actually a pair coherent state [29] with respect to the canonically transformed vector annihilation operator [16], although this pair coherent state does not have the restriction that the number difference of the two modes is fixed.

The most general superposition of harmonic oscillator coherent states is [22]

$$\int \frac{d^{2N} \vec{\alpha}}{\pi^N} f(\vec{\alpha}) |\vec{\alpha}\rangle \quad (8)$$

with  $|\vec{\alpha}\rangle = |\alpha_1\rangle \otimes |\alpha_2\rangle \otimes \cdots \otimes |\alpha_N\rangle$  the  $N$ -particle coherent state. The superposition (8) is entangled when it cannot be expressed as a product state in any representation.

We can introduce an entangled  $SU(2)$  coherent state as a generalization of (2). However, we make this analysis more general than necessary for studying qubits. We wish to treat general irreducible representations  $j$ , where  $j$  is the angular momentum parameter and can be integer or half-odd integer. The single qubit case corresponds to  $j = \frac{1}{2}$ . A pair of qubits can, of course, be treated as a single system: in this case we have one  $j = 0$  state (the singlet state) and  $j = 1$  states (the triplet states). The four states together constitute the Bell states (3).

### 2.2. Entangled $SU(2)$ coherent states

The  $SU(2)$  coherent state can be expressed as [4, 5, 31]

$$\begin{aligned} |j\gamma\rangle &\equiv R(\gamma)|jj\rangle \\ &= \exp[-\frac{1}{2}\theta(J_+ e^{-i\phi} - J_- e^{i\phi})] |jj\rangle \end{aligned}$$

$$= (1 + |\gamma|^2)^{-j} \sum_{m=0}^{2j} \binom{2j}{m}^{1/2} \gamma^m |jj - m\rangle \tag{9}$$

where  $\gamma = \exp(i\phi) \tan(\theta/2)$ ,  $R(\gamma)$  is the rotation operator, and  $J_-$  and  $J_+$  are lowering and raising operators of the  $su(2)$  Lie algebra, respectively. The  $su(2)$  generators  $J_{\pm}$  and  $J_z$  satisfy the  $su(2)$  commutation relations:

$$[J_+, J_-] = 2J_z \quad [J_z, J_{\pm}] = \pm J_{\pm}. \tag{10}$$

We can define an  $su(2)$  parity operator as

$$\Pi = (-1)^{\mathcal{M}} \quad \Pi^2 = 1 \quad \Pi^\dagger = \Pi. \tag{11}$$

Here  $\mathcal{M} = J_z + j$  is the ‘number’ operator such that

$$\mathcal{M}|j, m\rangle = (j + m)|j, m\rangle \quad 0 \leq j + m \leq 2j. \tag{12}$$

It is easy to see that

$$\Pi J_{\pm} \Pi = -J_{\pm} \quad \Pi J_z \Pi = J_z \tag{13}$$

which yields a new  $su(2)$  representation:

$$[\tilde{J}_+, \tilde{J}_-] = 2\tilde{J}_z \quad [\tilde{J}_z, \tilde{J}_{\pm}] = \pm \tilde{J}_{\pm} \tag{14}$$

where

$$\tilde{J}_+ = J_+ \Pi \quad \tilde{J}_- = \Pi J_- \quad \tilde{J}_z \equiv J_z. \tag{15}$$

We define a new  $SU(2)$  coherent state associated with the  $su(2)$  algebra (14) as

$$\begin{aligned} |j\gamma\rangle_{\Pi} &\equiv \tilde{R}(\gamma)|jj\rangle \\ &= \exp\left[-\frac{\theta}{2}(\tilde{J}_+ e^{-i\phi} - \tilde{J}_- e^{i\phi})\right]|jj\rangle. \end{aligned} \tag{16}$$

We call  $|j\gamma\rangle_{\Pi}$  the parity  $SU(2)$  coherent state, because the parity operator plays a central role in its definition. This term follows from that of the parity (harmonic oscillator) coherent state [16]. The state  $|j\gamma\rangle_{\Pi}$  is different from the  $SU(2)$  coherent state due to the nontrivial introduction of the  $su(2)$  parity operator  $\Pi$ .

The antinormally ordered rotation operator is

$$\tilde{R}(\gamma) = \exp(\gamma \tilde{J}_-)(1 + |\gamma|^2)^{-\tilde{J}_z} \exp(-\gamma^* \tilde{J}_+). \tag{17}$$

Using the above equation, we obtain

$$|j\gamma\rangle_{\Pi} = \frac{1}{\sqrt{2}} [e^{-i\frac{\pi}{4}} |j - i(-1)^{2j} \gamma\rangle + e^{i\frac{\pi}{4}} |ji(-1)^{2j} \gamma\rangle]. \tag{18}$$

The  $SU(2)$  parity coherent state is a superposition of two  $SU(2)$  coherent states with a phase difference  $\pi$ . It is similar to the parity harmonic oscillator coherent states [16, 32].

The general entangled  $SU(2)$  coherent state, analogous to the entangled coherent state (8), is discussed in appendix A. Here we introduce a specific  $SU(2)$  coherent state by employing the  $su(2)$  parity operator (11). Let us consider two independent  $su(2)$  Lie algebras. From these two algebras, we can define two new algebras:

$$\begin{aligned} [\tilde{J}_+^n, \tilde{J}_-^l] &= 2\delta_{nl} \tilde{J}_z^n & [\tilde{J}_z^n, \tilde{J}_{\pm}^l] &= \pm \delta_{nl} \tilde{J}_{\pm}^n \\ [\tilde{J}_+^n, \tilde{J}_-^l] &= 0 \end{aligned} \tag{19}$$

where

$$\tilde{J}_+^n = J_+^n \Pi \quad \tilde{J}_-^n = \Pi J_-^n \quad \tilde{J}_z^n \equiv J_z^n \quad \Pi = (-1)^{\mathcal{M}_1 + \mathcal{M}_2} \tag{20}$$

and  $n, l \in \{1, 2\}$ . It is easy to see that the parity operator  $\Pi$  is self-adjoint and satisfies  $\Pi^2 = 1$ . These two new  $su(2)$  representation are dependent on each other. The  $SU(2)$  coherent state of the original  $su(2)$  algebra is

$$|j\vec{\gamma}\rangle = R(\vec{\gamma})|jj\rangle_1 \otimes |jj\rangle_2 \tag{21}$$

where  $\vec{\gamma} \equiv (\gamma_1, \gamma_2)$ ,  $|j\vec{\gamma}\rangle \equiv |j\gamma_1\rangle_1 \otimes |j\gamma_2\rangle_2$ , and  $R(\vec{\gamma}) \equiv R_1(\gamma_1)R_2(\gamma_2)$ . It is important that both Hilbert spaces concerned in the entanglement are restricted to the same irreducible representation  $j$  for the entangled coherent states as we introduce them to be well defined for the  $j = \frac{1}{2}$  case. We consider entangled qubit states; hence the restriction to the same  $j$  is of particular value as well. However, the  $SU(2)$  coherent state for the two new  $su(2)$  representations is

$$\begin{aligned} |j\vec{\gamma}\rangle_\Pi &= \tilde{R}(\vec{\gamma})|jj\rangle_1 \otimes |jj\rangle_2 \\ &= \frac{1}{\sqrt{2}}(e^{-i\frac{\pi}{4}}|j - i\vec{\gamma}\rangle + e^{i\frac{\pi}{4}}|ji\vec{\gamma}\rangle) \end{aligned} \tag{22}$$

and is an entangled  $SU(2)$  coherent state-particle parity symmetry.

### 2.3. Entangled $SU(1, 1)$ coherent states

The generators of  $su(1, 1)$  Lie algebras,  $K_\pm$  and  $K_z$ , satisfy the commutation relations

$$[K_+, K_-] = -2K_z \quad [K_z, K_\pm] = \pm K_\pm. \tag{23}$$

By analogy to the  $su(2)$  case, we can define the  $su(1, 1)$  parity operator as

$$\Pi = (-1)^{\mathcal{N}} \quad \Pi^2 = 1 \quad \Pi^\dagger = \Pi. \tag{24}$$

where the ‘number’ operator  $\mathcal{N}$  is given by

$$\mathcal{N} = K_z - k \quad \mathcal{N}|kn\rangle = n|kn\rangle. \tag{25}$$

Here  $|kn\rangle (n = 0, 1, 2, \dots)$  is the complete orthonormal basis and  $k \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \dots\}$  is the Bargmann index labeling the irreducible representation ( $k(k - 1)$  is the value of the Casimir operator).

Using the  $su(1, 1)$  parity operator we can introduce a new  $su(1, 1)$  algebra with generators

$$\tilde{K}_+ = K_+\Pi \quad \tilde{K}_- = \Pi K_- \quad \tilde{K}_z \equiv K_z. \tag{26}$$

There are two distinct  $SU(1, 1)$  coherent states to consider.

**2.3.1. Entangled Perelomov  $SU(1, 1)$  coherent states.** The Perelomov coherent state of the  $su(1, 1)$  algebra is defined as [5, 30]

$$\begin{aligned} |k\eta\rangle_P &= S(\xi)|k0\rangle \\ &= \exp(\xi K_+ - \xi^* K_-)|k0\rangle \\ &= (1 - |\eta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)n!}} \eta^n |kn\rangle \end{aligned} \tag{27}$$

where  $\xi = r \exp(i\theta)$ ,  $\eta = \exp(i\theta) \tanh r$ ,  $\Gamma(x)$  is the gamma function and  $S(\xi)$  is the  $su(1, 1)$  displacement operator.

We define a new  $SU(1, 1)$  coherent state in association with the  $su(1, 1)$  parity operator:

$$\begin{aligned} |k\eta\rangle_{P\Pi} &= \tilde{S}(\xi)|k0\rangle \\ &= \exp(\xi \tilde{K}_+ - \xi^* \tilde{K}_-)|k0\rangle. \end{aligned} \tag{28}$$

It is found that the Perelomov  $SU(1, 1)$  coherent state is a nonlinear coherent state with the nonlinear function  $1/(\mathcal{N} + 2k)$  [33]. Therefore, the parity Perelomov  $SU(1, 1)$  coherent state is a nonlinear coherent state with the nonlinear function  $(-1)^{\mathcal{N}}/(\mathcal{N} + 2k)$ .

The normally ordered  $su(1, 1)$  displacement operator is

$$\tilde{S}(\xi) = \exp(\eta \tilde{K}_+) (1 - |\eta|^2)^{\tilde{K}_z} \exp(-\eta^* \tilde{K}_-) \tag{29}$$

with  $\eta = \frac{\xi}{|\xi|} \tanh(|\xi|)$ .

Using equation (29), we obtain

$$|k\eta\rangle_{P\Pi} = \frac{1}{\sqrt{2}} (e^{i\frac{\pi}{4}} |k - i\eta\rangle_P + e^{-i\frac{\pi}{4}} |ki\eta\rangle_P). \tag{30}$$

The parity Perelomov  $SU(1, 1)$  coherent states are superpositions of two Perelomov  $SU(1, 1)$  coherent states.

The general entangled  $SU(1, 1)$  coherent state is treated in appendix A, but here we consider the two-particle case. From the two  $su(1, 1)$  algebras, we define two new  $su(1, 1)$  algebras as

$$\begin{aligned} [\tilde{K}_+^n, \tilde{K}_-^l] &= -2\delta_{nl} \tilde{K}_z^n \\ [\tilde{K}_z^n, \tilde{K}_\pm^l] &= \pm\delta_{nl} \tilde{K}_\pm^l \quad [\tilde{K}_-^n, \tilde{K}_-^l] = 0 \end{aligned} \tag{31}$$

where

$$\tilde{K}_+^n = K_+^n \Pi \quad \tilde{K}_-^n = \Pi K_-^n \quad \tilde{K}_z^n \equiv K_z^n \quad \Pi = (-1)^{\mathcal{N}_1 + \mathcal{N}_2} \tag{32}$$

and  $n, l \in \{1, 2\}$ . The  $su(1, 1)$  parity operator  $\Pi$  is self-adjoint and satisfies  $\Pi^2 = 1$ .

The Perelomov  $SU(1, 1)$  coherent state for the two new  $su(1, 1)$  algebras is

$$\begin{aligned} |k\vec{\eta}\rangle_{P\Pi} &= \tilde{S}(\vec{\xi}) |k\vec{0}\rangle \\ &= \frac{1}{\sqrt{2}} (e^{i\frac{\pi}{4}} |k - i\vec{\eta}\rangle_P + e^{-i\frac{\pi}{4}} |ki\vec{\eta}\rangle_P) \end{aligned} \tag{33}$$

with  $\tilde{S}(\vec{\xi}) \equiv \tilde{S}_1(\xi_1) \tilde{S}_2(\xi_2)$ . The state  $|k\vec{\eta}\rangle_{P\Pi}$  is an entangled Perelomov  $SU(1, 1)$  coherent state. Again each Hilbert space is restricted to the same irrep  $k$ , similar to the restriction for the  $su(2)$  case.

**2.3.2. Entangled Barut–Girardello  $SU(1, 1)$  coherent states.** There is another coherent state of the  $su(1, 1)$  algebra known as the Barut–Girardello coherent state [26]. It is defined as the eigenstate of the lowering operator  $K_-$ :

$$K_- |k\eta\rangle_{BG} = \eta |k\eta\rangle_{BG} \tag{34}$$

and it can be expressed as [26]

$$|k\eta\rangle_{BG} = \sqrt{\frac{|\eta|^{2k-1}}{I_{2k-1}(2|\eta|)}} \sum_{n=0}^{\infty} \frac{\eta^n}{\sqrt{n! \Gamma(n + 2k)}} |kn\rangle \tag{35}$$

where  $I_\nu(x)$  is the modified Bessel function of the first kind. The Perelomov coherent state is defined with respect to the displacement operator formalism, whereas the Barut–Girardello coherent state is defined with respect to the ladder-operator formalism. Thus, we define the parity Barut–Girardello  $SU(1, 1)$  coherent state as

$$\tilde{K}_- |k\eta\rangle_{BG\Pi} = (-1)^{\mathcal{N}} K_- |k\eta\rangle_{BG\Pi} = \eta |k\eta\rangle_{BG\Pi}. \tag{36}$$

The state  $|k\eta\rangle_{\text{BG}\Pi}$  is a nonlinear coherent state with the nonlinear function  $(-1)^{\mathcal{N}}$ . From the general expression of an  $SU(1, 1)$  nonlinear coherent state [33], we obtain the expression of the state  $|k\eta\rangle_{\text{BG}\Pi}$  as

$$|k\eta\rangle_{\text{BG}\Pi} = \frac{1}{\sqrt{2}}(e^{i\frac{\pi}{4}}|k - i\eta\rangle_{\text{BG}} + e^{-i\frac{\pi}{4}}|ki\eta\rangle_{\text{BG}}). \tag{37}$$

Eigenstates of operators  $\tilde{K}_-^n$ , for  $n \in \{1, 2\}$ , are constructed as

$$\begin{aligned} |k\vec{\eta}\rangle_{\text{BG}\Pi} &\sim \exp\left(\frac{\eta_1}{\mathcal{N}_1 + 2k - 1}\tilde{K}_+^1\right)\exp\left(\frac{\eta_2}{\mathcal{N}_2 + 2k - 1}\tilde{K}_+^2\right)|k\vec{0}\rangle \\ &= \frac{1}{\sqrt{2}}(e^{i\frac{\pi}{4}}|k - i\vec{\eta}\rangle_{\text{BG}} + e^{-i\frac{\pi}{4}}|ki\vec{\eta}\rangle_{\text{BG}}) \end{aligned} \tag{38}$$

which is the entangled Barut–Girardello  $SU(1, 1)$  coherent states. Here we have used the exponential form of the unnormalized Barut–Girardello coherent state [33]:

$$|k\eta\rangle_{\text{BG}} \sim \exp\left(\frac{\eta}{\mathcal{N} + 2k - 1}K_+\right)|k0\rangle. \tag{39}$$

The generalization of (22), (31) and (38) to multiparticle entangled  $SU(2)$  and  $SU(1, 1)$  coherent states is treated in appendix A.

### 3. Nonlinear $su(2)$ and $su(1, 1)$ coherent states

#### 3.1. $SU(2)$ case

The  $su(2)$  parity operator  $\Pi = (-1)^{\mathcal{M}}$  is a special case of the unitary operator

$$U(\mathcal{M}) = \exp[-i\vartheta(\mathcal{M})]. \tag{40}$$

Here  $\vartheta(\mathcal{M})$  is a nonsingular function of  $\mathcal{M}$ . The operators

$$\bar{J}_+ = J_+U(\mathcal{M}) \quad \bar{J}_- = U^\dagger(\mathcal{M})J_- \quad \bar{J}_z \equiv J_z \tag{41}$$

satisfy the  $su(2)$  commutation relations (10). Then we can define the  $SU(2)$  coherent state corresponding to this  $su(2)$  algebra as

$$\begin{aligned} |\vartheta; j\gamma\rangle &\equiv \bar{R}(\gamma)|jj\rangle \\ &= \exp[-\frac{1}{2}\theta(\bar{J}_+e^{-i\phi} - \bar{J}_-e^{i\phi})]|jj\rangle. \end{aligned} \tag{42}$$

From equation (17), we obtain

$$|\vartheta; j\gamma\rangle = (1 + |\gamma|^2)^{-j} \sum_{m=0}^{2j} \binom{2j}{m}^{1/2} \gamma^m \exp\left[i \sum_{n=1}^m \vartheta(2j - n)\right]|jj - m\rangle \tag{43}$$

where we have used the relation

$$[f(\mathcal{M})J_-]^m = (J_-)^m \prod_{n=1}^m f(\mathcal{M} - n). \tag{44}$$

If we choose  $\vartheta(\mathcal{M}) = \pi\mathcal{M}$ , equation (43) reduces to (18) as expected. The state  $|\vartheta; j\gamma\rangle$  is a nonlinear  $SU(2)$  coherent state if  $\vartheta(\mathcal{M})$  is a nonlinear function of  $\mathcal{M}$ .



3.2.  $SU(1, 1)$  case

By analogy with the  $su(2)$  case (41), we define

$$\bar{K}_+ = K_+ V(\mathcal{N}) \quad \bar{K}_- = V^\dagger(\mathcal{N}) K_- \quad \bar{K}_z \equiv K_z \tag{45}$$

with  $V(\mathcal{N}) = \exp[-i\varphi(\mathcal{N})]$ . These operators satisfy the  $su(1, 1)$  commutation relations (23). The Perelomov  $SU(1, 1)$  coherent state corresponding to this  $su(1, 1)$  algebra is

$$\begin{aligned} |\varphi; k\eta\rangle_P &= \bar{S}(\xi)|k0\rangle \\ &= \exp(\xi \bar{K}_+ - \xi^* \bar{K}_-)|k0\rangle. \end{aligned} \tag{46}$$

As mentioned in section 2, the Perelomov  $SU(1, 1)$  coherent state  $|k\eta\rangle_P$  is a nonlinear coherent state with the nonlinear function  $1/(\mathcal{N} + 2k)$ . Therefore, the state  $|\varphi; k\eta\rangle_P$  is a nonlinear coherent state with the nonlinear function

$$\frac{e^{i\varphi(\mathcal{N})}}{\mathcal{N} + 2k}. \tag{47}$$

From the general expression for an  $su(1, 1)$  nonlinear coherent [33], we obtain

$$|\varphi; k\eta\rangle_P = (1 - |\eta|^2)^k \sum_{n=0}^{\infty} \sqrt{\frac{\Gamma(2k+n)}{\Gamma(2k)n!}} \eta^n \exp\left[-i \prod_{m=0}^{n-1} \varphi(m)\right] |kn\rangle. \tag{48}$$

In the derivation of the above equation, we have used the relation

$$[K_+ f(\mathcal{N})]^m = (K_+)^m \prod_{n=1}^m f(\mathcal{N} + n - 1). \tag{49}$$

The nonlinear Barut–Girardello coherent state is defined as

$$\begin{aligned} \bar{K}_-|\varphi; k\eta\rangle_{BG} &= \exp[i\varphi(\mathcal{N})]K_-|\varphi; k\eta\rangle_{BG} \\ &= \eta|\varphi; k\eta\rangle_{BG}. \end{aligned} \tag{50}$$

The state  $|\varphi; k\eta\rangle_{BG}$  is also a nonlinear coherent state with the nonlinear function  $\exp[i\varphi(\mathcal{N})]$ . The expansion of this state yields

$$|\varphi; k\eta\rangle_{BG} = \sqrt{\frac{|\eta|^{2k-1}}{I_{2k-1}(2|\eta|)}} \sum_{n=0}^{\infty} \frac{1}{\sqrt{\Gamma(2k+n)n!}} \eta^n \exp\left[-i \prod_{m=0}^{n-1} \varphi(m)\right] |kn\rangle. \tag{51}$$

If we choose  $\varphi(\mathcal{N}) = \pi\mathcal{N}$ , equations (48) and (51) reduce to (29) and (37), respectively, as expected.

4. Representation of entangled  $su(2)$  and  $su(1, 1)$  coherent states in Fock space

4.1. Entangled binomial states

It is well known that the operators

$$J_+ = a^\dagger \sqrt{M - N} \quad J_- = \sqrt{M - N} a \quad J_z = N - M/2 \tag{52}$$

generate the  $su(2)$  algebra via the Holstein–Primakoff representation [34] in the spin- $\frac{M}{2}$  representation. Here  $N = a^\dagger a$  is the number operator. The vacuum  $|0\rangle$  is the lowest-weight state

$$J_-|0\rangle = 0 \quad J_z|0\rangle = -(M/2)|0\rangle. \tag{53}$$

Then we define the  $SU(2)$  coherent state in the displacement-operator form:

$$\begin{aligned} |M\eta\rangle &= \exp(\zeta J_+ - \zeta^* J_-)|0\rangle \\ &= \exp\left(\zeta a^\dagger \sqrt{M - N} - \zeta^* \sqrt{M - N} a\right)|0\rangle \end{aligned} \tag{54}$$

where we choose  $\zeta = \exp(i\theta) \arctan(|\eta|/\sqrt{1-|\eta|^2})$ ,  $\eta = |\eta| \exp(i\theta)$ . The normally ordered form of the displacement operator is

$$\exp(\zeta J_+ - \zeta^* J_-) = \exp(\tilde{\eta} J_+) (1 + |\tilde{\eta}|^2)^{J_z} \exp(-\tilde{\eta}^* J_-) \tag{55}$$

where  $\tilde{\eta} = \zeta/|\zeta| \tan |\zeta| = \eta/\sqrt{1-|\eta|^2}$ . By use of (55), we obtain

$$|M\eta\rangle = (1 - |\eta|^2)^{M/2} \sum_{n=0}^M \binom{M}{n}^{1/2} \left( \frac{\eta}{\sqrt{1-|\eta|^2}} \right)^n |n\rangle \tag{56}$$

which is the binomial state [35–40]. Therefore, the binomial state is a special type of  $SU(2)$  coherent state via the Holstein–Primakoff representation. From equations (11) and (52), we see that the parity operator is  $(-1)^N$ . By analogy with the discussion in section 2, we can obtain the entangled binomial states of the type (22). Now we show how to generate entangled binomial states in a particular Hamiltonian system.

The entangled coherent states can be created using an ideal Kerr nonlinearity with three nonlinear media elements [22]. We show that the entangled binomial state can also be generated in this system. By an appropriate arrangement of the three nonlinear elements, the effective ideal Kerr transformation with two input fields, 1 and 2, is given by [22]

$$S_{12}(\chi) = \exp(-i\chi a_1^\dagger a_1 a_2^\dagger a_2). \tag{57}$$

We assume that the initial state is a product of binomial states  $|M\vec{\eta}\rangle = |M\eta_1\rangle_1 \otimes |M\eta_2\rangle_2$ . For  $\chi = \pi$ , the resulting output state is

$$\frac{1}{2}(|M\eta_1\rangle_1 + |M - \eta_1\rangle_1) \otimes |M\eta_2\rangle_2 + (|M\eta_1\rangle_1 - |M - \eta_1\rangle_1) \otimes |M - \eta_2\rangle_2. \tag{58}$$

This state contains both even and odd binomial states [41].

#### 4.2. Entangled negative binomial states

The generators of the  $su(1, 1)$  algebra, obtained via the Holstein–Primakoff realization of the discrete irreducible representation with Bargmann index  $M/2$ , are

$$K_+ = a^\dagger \sqrt{M+N} \quad K_- = \sqrt{M+N} a \quad K_z = N + M/2 \tag{59}$$

and the vacuum  $|0\rangle$  is the lowest-weight state

$$K_-|0\rangle = 0 \quad K_z|0\rangle = M/2|0\rangle. \tag{60}$$

The corresponding  $SU(1, 1)$  coherent state is

$$|M\eta\rangle^- = \exp(\xi K_+ - \xi^* K_-)|0\rangle. \tag{61}$$

Here  $\xi = \exp(i\theta) \operatorname{arctanh}|\eta|$ . Using equation (29), we obtain

$$|M\eta\rangle^- = (1 - |\eta|^2)^{M/2} \sum_{n=0}^{\infty} \binom{M+n-1}{n}^{1/2} \eta^n |n\rangle \tag{62}$$

which is the negative binomial state [42–45]. Therefore, the negative binomial state is a special type of  $SU(1, 1)$  Perelomov coherent state. The parity operator is  $(-1)^N$  and the entangled negative binomial states of the type (31) can be obtained. We do not write these entangled states explicitly here.

Under the transformation  $S_{12}(\pi)$ , the input negative binomial state  $|M, \vec{\eta}\rangle^-$  will be transformed into the entangled negative binomial states

$$\frac{1}{2}(|M\eta_1\rangle_1^- + |M - \eta_1\rangle_1^-) \otimes |M\eta_2\rangle_2^- + (|M\eta_1\rangle_1^- - |M - \eta_1\rangle_1^-) \otimes |M - \eta_2\rangle_2^-. \tag{63}$$

This state contains both even and odd negative binomial states [46,47].

4.3. Contraction of  $su(2)$  and  $su(1, 1)$  algebra

The binomial distribution tends to the Poisson distribution in a certain limit. Let  $M \rightarrow \infty, |\eta| \rightarrow 0$  in such a way that the product  $|\eta|^2 M = |\alpha|^2$  is fixed. In this limit, the binomial distribution of the binomial state tends to the Poisson distribution  $\exp(-|\alpha|^2)|\alpha|^{2n}/n!$ , and the binomial state tends to the ordinary coherent state:

$$|M\eta\rangle \rightarrow \exp(-|\alpha|^2/2) \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle. \tag{64}$$

Here we have used the relation

$$(1 - |\eta|^2)^M \rightarrow \exp(-|\alpha|^2). \tag{65}$$

This limit can also be visualized as a contraction of the  $su(2)$  algebra into the Heisenberg–Weyl algebra [48]:

$$|\eta|J_+ \rightarrow |\alpha|a^\dagger \quad |\eta|J_- \rightarrow |\alpha|a. \tag{66}$$

Thus, (56) tends to the coherent state

$$|M\eta\rangle \rightarrow \exp[\alpha a^\dagger - \alpha^* a]|0\rangle. \tag{67}$$

In this limit the entangled binomial state (58) reduces to the entangled coherent state [22]

$$\frac{1}{2}(|\alpha_1\rangle_1 + |-\alpha_1\rangle_1) \otimes |\alpha_2\rangle_2 + (|\alpha_1\rangle_1 - |-\alpha_1\rangle_1) \otimes |-\alpha_2\rangle_2 \tag{68}$$

which has applications as qubits in quantum computation [49, 50].

In the limit described above, the negative binomial states reduce to coherent states, the  $su(1, 1)$  algebra contracts into the Heisenberg–Weyl algebra [48], and the entangled negative binomial states (63) reduce to the entangled harmonic oscillator coherent states (68).

4.4. Entangled squeezed states

The amplitude-squared  $su(1, 1)$  algebra is realized by

$$K_+ = \frac{1}{2}a^{+2} \quad K_- = \frac{1}{2}a^2 \quad K_z = \frac{1}{2}(N + \frac{1}{2}). \tag{69}$$

The representation on the usual Fock space is completely reducible and decomposes into a direct sum of the even Fock space ( $S_0$ ) and odd Fock space ( $S_1$ ):

$$S_i = \text{span}\{|n\rangle_i \equiv |2n + i\rangle | n = 0, 1, 2, \dots\} \tag{70}$$

for  $i \in \{0, 1\}$ .

Representations on  $S_i$  can be written as

$$\begin{aligned} K_+|n\rangle_i &= \sqrt{(n+1)(n+i+1/2)}|n+1\rangle_i \\ K_-|n\rangle_i &= \sqrt{n(n+i-1/2)}|n-1\rangle_i \\ K_0|n\rangle_i &= (n+i/2+1/4)|n\rangle_i. \end{aligned} \tag{71}$$

The Bargmann index  $k = 1/4$  ( $3/4$ ) for even (odd) Fock space. We see that the Perelomov  $SU(1, 1)$  coherent states in even (odd) Fock space are squeezed vacuum (first Fock) states:

$$\begin{pmatrix} |\eta\rangle_V \\ |\eta\rangle_F \end{pmatrix} = \exp\left(\frac{\xi}{2}a^{+2} - \frac{\xi^*}{2}a^2\right) \begin{pmatrix} |0\rangle \\ |1\rangle \end{pmatrix}. \tag{72}$$

From (24) and (69), the corresponding parity operator is  $(-1)^{N/2}$  in even Fock space and  $(-1)^{(N-1)/2}$  in odd Fock space. We consider two modes  $a_1$  and  $a_2$ . Then from (31) and

through the amplitude-squared  $su(1, 1)$  realization, we obtain the entangled squeezed vacuum states and entangled squeezed first Fock states as

$$\begin{aligned} |\bar{\eta}\rangle_V &= \frac{1}{\sqrt{2}}(e^{i\frac{\pi}{4}}|-\bar{i}\bar{\eta}\rangle_V + e^{-i\frac{\pi}{4}}|i\bar{\eta}\rangle_V) \\ |\bar{\eta}\rangle_F &= \frac{1}{\sqrt{2}}(e^{i\frac{\pi}{4}}|-\bar{i}\bar{\eta}\rangle_F + e^{-i\frac{\pi}{4}}|i\bar{\eta}\rangle_F). \end{aligned} \tag{73}$$

The state (73) is a special case of the entangled squeezed coherent state [14] for zero amplitude and reduces to the superposition of two squeezed vacuum states [27].

In Fock space, we have obtained entangled binomial states, entangled negative binomial states and entangled squeezed states. They are special cases of entangled  $SU(2)$  coherent states or entangled  $SU(1, 1)$  coherent states.

### 5. Generation of the entangled coherent states

A superposition of two distinct  $su(2)$  coherent states can be generated by the Hamiltonian system for the nonlinear rotator [31, 51]

$$H_j = \omega J_z + \frac{\lambda}{2j} J_z^2 (\hbar = 1) \tag{74}$$

where  $\omega$  is the linear precession frequency and  $\lambda \in \mathbb{R}$  is a constant. Let the initial state be the  $SU(2)$  coherent state  $|j, \gamma\rangle$ . At time  $t_j = \pi j/\lambda$ , the coherent state has evolved into the superposition state

$$\exp(-iH_j t_j)|j\gamma\rangle = \frac{1}{\sqrt{2}}(e^{-i\frac{\pi}{4}}|j\bar{\gamma}\rangle + e^{i\frac{\pi}{4}}(-1)^j|j - \bar{\gamma}\rangle) \tag{75}$$

with  $\bar{\gamma} = \exp(i\omega\pi j/\lambda)\gamma$ . A superposition of two coherent states, separated by a phase  $\pi$ , has been generated from the initial  $SU(2)$  coherent state. The phase difference of the coefficients in (75) depends on whether  $j$  is odd or even. Let  $\bar{\gamma} \rightarrow -i(-1)^{2j}\gamma$  and  $j$  be even; the above state is just the parity  $SU(2)$  coherent state  $|j\gamma\rangle_\Pi$ .

Gerry [52] and Bužek [53] have studied the dynamics of the nonlinear oscillator with the Hamiltonian  $H$  written as

$$H = \omega a^\dagger a + \frac{\lambda}{2} a^{\dagger 2} a^2 = \omega K_z + \lambda K_+ K_- \tag{76}$$

up to constant terms. Here  $K_z$  and  $K_\pm$  are generators of the amplitude-squared  $su(1, 1)$  algebra (69). At time  $t = \pi/2\lambda$ , the initial coherent state has evolved into the superposition state

$$\frac{1}{\sqrt{2}}[e^{-i\frac{\pi}{4}}|k\bar{\eta}\rangle_\Sigma + e^{i\frac{\pi}{4}}|k - \bar{\eta}\rangle_\Sigma] \tag{77}$$

with  $\Sigma$  either P for Perelomov coherent states or BG for Barut–Girardello coherent states, and

$$\bar{\eta} = \exp\{-i\pi[\omega + (2k - 1)\lambda]/(2\lambda)\}. \tag{78}$$

The initial  $SU(1, 1)$  coherent state has been chosen as  $|k\eta\rangle_\Sigma$ . Let  $\bar{\eta} \rightarrow i\eta$ ; the above superposition states are then just the parity coherent states  $|k\eta\rangle_{\Sigma\Pi}$ .

Now we consider how to generate the entangled  $SU(2)$  and  $SU(1, 1)$  coherent states. By analogy with the generation of entangled coherent states of the harmonic oscillator [16], the two Hamiltonians are given by

$$H_2 = \chi_1 \mathcal{M}_1^2 + \chi_2 \mathcal{M}_2^2 + \chi_3 \mathcal{M}_1 \mathcal{M}_2 \tag{79}$$

$$H_{11} = \lambda_1 \mathcal{N}_1^2 + \lambda_2 \mathcal{N}_2^2 + \lambda_3 \mathcal{N}_1 \mathcal{N}_2. \tag{80}$$

We assume that the initial state of the Hamiltonian system  $H_2$  is  $|j - i\vec{\gamma}\rangle$  and the time is  $t = \pi/(2\chi_1)$ ,  $\chi_2 = \chi_1$ ,  $\chi_3 = 2\chi_1$ . Then the initial state evolves into the entangled  $SU(2)$  coherent state  $|j\vec{\gamma}\rangle_\Pi$  ( $j$  is an integer). Similarly, we assume that the initial state of the Hamiltonian system  $H_{11}$  is  $|ki\vec{\eta}\rangle_\Sigma$  and the time  $t = \pi/(2\lambda_1)$ ,  $\lambda_2 = \lambda_1$ ,  $\lambda_3 = 2\lambda_1$ , for  $\Sigma$  either P or BG. The initial state evolves into the entangled  $SU(1, 1)$  coherent state  $|k\vec{\eta}\rangle_{\Sigma\Pi}$ .

It is interesting if we choose the parameters  $\chi_1 = \chi_2 = 0$ ,  $t = \pi/\chi_3$ . The initial state  $|j\vec{\gamma}\rangle$  will evolve into the following entangled  $SU(2)$  coherent state:

$$\begin{aligned} & \frac{1}{2}\{[|j\gamma_1\rangle_1 + (-1)^{2j}|j - \gamma_1\rangle_1] \otimes |j\gamma_2\rangle_2 + [(-1)^{2j}|j\gamma_1\rangle_1 - |j - \gamma_1\rangle_1] \otimes |j - \gamma_2\rangle_2\} \\ &= \frac{1}{2}\{|j\gamma_1\rangle_1 \otimes [|j\gamma_2\rangle_2 + (-1)^{2j}|j - \gamma_2\rangle_2] + |j - \gamma_1\rangle_1 \\ & \quad \otimes [(-1)^{2j}|j\gamma_2\rangle_2 - |j - \gamma_2\rangle_2]\}. \end{aligned} \quad (81)$$

The entangled coherent state (81) includes the  $SU(2)$  ‘cat’ states [31, 54]. We assume that the initial state of the Hamiltonian system  $H_{11}$  is  $|k\vec{\eta}\rangle$ . Here the subscript  $\Sigma$  has been omitted as the analysis is the same for the Perelomov coherent states and the Barut–Girardello coherent states. At time  $t = \pi/\lambda_3$  and  $\lambda_1 = \lambda_2 = 0$ , the resultant state is

$$\frac{1}{2}\{(|k\eta_1\rangle_1 + |k - \eta_1\rangle_1) \otimes |k\eta_2\rangle_2 + (|k\eta_1\rangle_1 - |k - \eta_1\rangle_1) \otimes |k - \eta_2\rangle_2\} \quad (82)$$

which includes the  $SU(1, 1)$  ‘cat’ state [54].

## 6. Degree of entanglement

### 6.1. Bell inequality

A standard example of entanglement of two-particle nonorthogonal states is given by [13]

$$|\Psi\rangle = \mu|\alpha\rangle_1 \otimes |\beta\rangle_2 + \nu|\gamma\rangle_1 \otimes |\delta\rangle_2 \quad (83)$$

where  $|\alpha\rangle_1$  and  $|\gamma\rangle_1$  are nonorthogonal states of system 1 and similarly for  $|\beta\rangle_2$  and  $|\delta\rangle_2$  of system 2. We also assume that the states  $|\alpha\rangle_1$  and  $|\gamma\rangle_1$  are linearly independent, as are  $|\beta\rangle_2$  and  $|\delta\rangle_2$ . The entangled  $SU(2)$  and  $SU(1, 1)$  coherent states obtained in the previous section are special cases of these entangled nonorthogonal states  $|\Psi\rangle$ . The state  $|\Psi\rangle$  is a pure state whose density matrix is  $\rho_{12} = |\Psi\rangle\langle\Psi|$ .

From this information, the reduced density matrices  $\rho_1 = \text{Tr}_2(\rho_{12})$  and  $\rho_2 = \text{Tr}_1(\rho_{12})$  for each subsystem can be obtained, and the two eigenvalues of  $\rho_1$  are given by [13]

$$\begin{aligned} \lambda_{\pm} &= \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - 4|\mu\nu\mathcal{A}_1\mathcal{A}_2|^2} \\ \mathcal{A}_1 &= \sqrt{1 - |\langle\alpha|\gamma\rangle_1|^2} \quad \mathcal{A}_2 = \sqrt{1 - |\langle\delta|\beta\rangle_2|^2}. \end{aligned} \quad (84)$$

The two eigenvalues of  $\rho_2$  are identical to those of  $\rho_1$ . The corresponding eigenvectors of  $\rho_1$  are  $|\pm\rangle_1$ , and the corresponding orthonormal eigenvectors of  $\rho_2$  are denoted by  $|\pm\rangle_2$ . The general theory of Schmidt decomposition [55–57] implies that the state  $|\Psi\rangle$  can be expressed in the Schmidt form

$$|\Psi\rangle = c_-|-\rangle_1|-\rangle_2 + c_+|+\rangle_1|+\rangle_2 \quad (85)$$

where  $|c_{\pm}|^2 = \lambda_{\pm}$ ,  $|c_+|^2 + |c_-|^2 = 1$ . The two-system entangled state (85) violates a Bell inequality. More specifically, we choose a Hermitian operator  $\hat{\Theta}_i$  for each subsystem such that the eigenvalues are  $\pm 1$ . The general form for such an operator is

$$\hat{\Theta}_i(\lambda_i, \varphi_i) = \cos \lambda_i (|+\rangle_i \langle +| - |-\rangle_i \langle -|) + \sin \lambda_i (e^{i\varphi_i} |+\rangle_i \langle -| + e^{-i\varphi_i} |-\rangle_i \langle +|). \quad (86)$$

The Bell operator is defined as [58]

$$\hat{B} = \hat{\Theta}_1 \hat{\Theta}_2 + \hat{\Theta}_1' \hat{\Theta}_2' + \hat{\Theta}_1' \hat{\Theta}_2 - \hat{\Theta}_1 \hat{\Theta}_2' \quad (87)$$

for  $\hat{\Theta}_i \equiv \hat{\Theta}_i(\lambda_i, \varphi_i)$ ,  $\hat{\Theta}'_i \equiv \hat{\Theta}_i(\lambda'_i, \varphi'_i)$ .

For the choices

$$\begin{aligned} \lambda_1 &= 0 & \lambda'_1 &= \pi/2 \\ \lambda_2 &= -\lambda'_2 = \cos^{-1}[1 + |2c_+c_-|^2]^{-1/2} \\ \varphi_1 + \varphi_2 &= \varphi'_1 + \varphi'_2 = \varphi_+ - \varphi_- \end{aligned} \tag{88}$$

where  $\varphi_{\pm}$  are the phases of  $c_{\pm}$  in (85), the expectation value of the Bell operator for the state  $|\Psi\rangle$  (6.1) is [13]

$$B = \langle \Psi | \hat{B} | \Psi \rangle = 2\sqrt{1 + 4\lambda_+\lambda_-} > 2. \tag{89}$$

The degree of violation depends on the values of  $\lambda_{\pm}$ , but a violation always occurs provided that the pure state is entangled.

Several entangled coherent states are obtained in the previous sections. Here we only consider two examples. The first example is the entangled  $SU(2)$  coherent state  $|j\vec{\gamma}\rangle_{\Pi}$  (22). For simplicity  $\gamma_0 = \gamma_1 = \gamma_2$  is introduced. The eigenvalues of the reduced density matrices  $\rho_1$  and  $\rho_2$  are

$$\lambda_{\pm} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \left[ 1 - \left( \frac{1 - |\gamma_0|^2}{1 + |\gamma_0|^2} \right)^{4j} \right]^2}. \tag{90}$$

Then the expectation value of the Bell operator for the entangled coherent state  $|j\vec{\gamma}\rangle_{\Pi}$  is given by

$$B = 2 \sqrt{1 + \left[ 1 - \left( \frac{1 - |\gamma_0|^2}{1 + |\gamma_0|^2} \right)^{4j} \right]^2}. \tag{91}$$

The second example is the entangled Perelomov  $SU(1, 1)$  coherent state  $|k\vec{\eta}\rangle_{P\Pi}$ . The corresponding eigenvalues and expectation value of the Bell operator are obtained as

$$\lambda_{\pm} = \frac{1}{2} \pm \frac{1}{2} \sqrt{1 - \left[ 1 - \left( \frac{1 - |\eta_0|^2}{1 + |\eta_0|^2} \right)^{4k} \right]^2} \tag{92}$$

$$B = 2 \sqrt{1 + \left[ 1 - \left( \frac{1 - |\eta_0|^2}{1 + |\eta_0|^2} \right)^{4k} \right]^2}. \tag{93}$$

Here we have chosen  $\eta_0 = \eta_1 = \eta_2$ .

### 6.2. Entropy

The entropy  $S$  of a quantum state described by density operator  $\rho$  is defined by [59–61]

$$S = -\text{Tr}(\rho \ln \rho). \tag{94}$$

The entropy defined above is zero for a pure state and positive for a mixed state. If we consider two systems, the entropy of system 1(2) is determined via the reduced density operator

$$S_{1(2)} = -\text{Tr}_{1(2)}(\rho_{1(2)} \ln \rho_{1(2)}). \tag{95}$$

In 1970 Araki and Lieb proved the following inequality [59]:

$$|S_1 - S_2| \leq S \leq S_1 + S_2. \tag{96}$$

One consequence of this inequality is that, if the total system is in a pure state,  $S_1 = S_2$ . From an information theory point of view, the entropy can be regarded as the amount of uncertainty

contained within the density operator. We can use the index of correlation  $I_c$  as the amount of information lost in the tracing procedure [61]:

$$I_c = S_1 + S_2 - S. \quad (97)$$

For the pure state  $|\Psi\rangle$ , the index of correlation is

$$I_c = 2S_1 = -2(\lambda_+ \ln \lambda_+ + \lambda_- \ln \lambda_-). \quad (98)$$

If  $I_c = 0$ , the two subsystems are in a pure state and disentangled. The combination of (90)–(93) and (98) gives the index of correlation of entangled  $SU(2)$  and  $SU(1, 1)$  coherent states.

### 6.3. Discussion

We now discuss two limit cases for certain parameters. When  $|\gamma_0|(|\eta_0|) \rightarrow 0$  or  $\infty$ ,  $B = 2$ , i.e., the entangled coherent states become product states and disentangled. When  $|\gamma_0|(|\eta_0|) = 1$ ,  $B = 2\sqrt{2}$ , and the entangled coherent states are maximally entangled states. For this case, the two states  $|j\gamma_0\rangle$  and  $|j - \gamma_0\rangle$  are orthogonal and the states  $|k\eta_0\rangle$  and  $|k - \eta_0\rangle$  become orthogonal.

When  $|\gamma_0|(|\eta_0|) \rightarrow 0$  or  $\infty$ ,  $\lambda_+ = 1$  and  $\lambda_- = 0$ , then the index of correlation is zero, as we expect. The entangled states become product states. When  $|\gamma_0|(|\eta_0|) = 1$ ,  $\lambda_{\pm} = 1/2$ , then the index of correlation is  $2 \ln 2$ , which results from the orthogonality of the states  $|j \pm \gamma_0\rangle$  ( $|\gamma_0| = 1$ ) or  $|k \pm \eta_0\rangle$  ( $|\eta_0| = 1$ ). An index of correlation is maximum for  $I_c = 2 \ln 2$  [61].

The maximum entanglement of  $SU(2)$  coherent states  $|j\vec{\gamma}\rangle_{\Pi}$  for  $\vec{\gamma} = (\gamma_0, \gamma_0)$  and  $|\gamma_0| = 1$  can be understood as follows. From the expression for  $\gamma = \exp(i\phi) \tan(\theta/2)$ , we can see that  $|\gamma| = 1$  corresponds to the set of all points on the equator ( $\theta = \pi/2$ ) of the Poincaré sphere, which is used to represent the pure states for the  $SU(2)/U(1)$  coset space. The states corresponding to  $|j\gamma_0 = 1\rangle$  and  $|j\gamma_0 = -1\rangle$  are antipodal; that is, one state is represented by a point at the intersection of the equator and the longitudinal line  $\phi = 0$ , and the other state is represented by the point at the intersection of the equator and the longitudinal line  $\phi = \pi$ . We can thus employ the notation of equation (1), but with an appropriate  $SU(2)$  rotation (and allow for arbitrary  $j$ ) by replacing  $|j\gamma_0 = 1\rangle$  by  $|1\rangle$  and  $|j\gamma_0 = -1\rangle$  by  $|0\rangle$ . For  $j = 1/2$ , we recover the qubit case (1). The entangled state for  $j = 1/2$  is just

$$2^{-1/2}[\exp(i\pi/4)|11\rangle + \exp(-i\pi/4)|00\rangle]. \quad (99)$$

This state is a form of Bell state (3) and the reason for  $B$  and  $I_c$  being maximal is evident. A similar analysis can be employed for the entangled  $SU(1, 1)$  coherent states represented by points on the Lobachevsky plane, which is the manifold for the coset space  $SU(1, 1)/U(1)$ .

## 7. Conclusions

We have introduced entangled  $SU(2)$  and  $SU(1, 1)$  coherent states. The general form for these entangled coherent states, which also incorporates entangled harmonic oscillator coherent states in the formalism, is given in the appendix, but the main concern here is with two-particle coherent states. The two-particle coherent states present a diverse range of interesting phenomena, as we have shown. Aside from the mathematical elegance of entangled  $SU(2)$  and  $SU(1, 1)$  coherent states, we have also applied these states to current topics of research, namely quantum information (specifically qubits) and nonclassical states of light (specifically squeezed vacuum states).

We have explored several aspects of coherent states. Aside from defining these states, we have employed the entangled binomial states to establish a rigorous contraction from entangled  $SU(2)$  coherent states to entangled harmonic oscillator coherent states. Two measures of

entanglement for pure states, the Bell operator approach and the index of correlation, have been used to quantify the degree of entanglement. As the entanglement is generally between non-orthogonal states, the degree of entanglement ranges from no entanglement (product state) to being maximally entangled for various parametric choices.

The generation of entangled coherent states have been treated here by a Hamiltonian evolution which is a generalization of  $J_z^2$  and  $K_z^2$  nonlinear evolution for multiparticle systems. However, such evolutions are extremely sensitive to environmental-induced decoherence. Other methods for generating entangled coherent states could be considered, but the nonlinear evolution considered here illustrated one possible approach to producing these entangled states.

These results can be generalized in various ways which would be of interest. One generalization is to entangled generalized coherent states, with generalized coherent states of the type treated by Perelomov [5]. Another intriguing generalization is to entangled  $SU(2)$  and  $SU(1, 1)$  coherent states for the respective Hilbert spaces, not restricted to the same irreducible representations. For example, one could consider entanglement between spin- $\frac{1}{2}$  states (qubits) and spin-1 states (qutrits). Or one could consider entanglement of  $SU(1, 1)$  coherent states with  $k = 1/4$  (single-mode squeezed vacuum states) and  $SU(1, 1)$  coherent states with  $k = 3/4$  (two-mode squeezed vacuum states). These ideas are topics for future research.

**Appendix. Most general form of entangled  $SU(2)$  and  $SU(1, 1)$  coherent states**

The general form of entangled  $SU(2)$  and  $SU(1, 1)$  coherent states can be written as

$$\int d\mu(\vec{\xi}) f(\vec{\xi}) |l\vec{\xi}\rangle \tag{A.1}$$

where  $l = j$  and  $\vec{\xi} = \vec{\gamma}$  for  $SU(2)$  coherent states, and  $l = k$  and  $\vec{\xi} = \vec{\eta}$  for either Perelomov or Barut–Girardello  $SU(1, 1)$  coherent states. We can also obtain the entangled coherent state (2.5) via the same expression by replacing  $\vec{\xi}$  by  $\vec{\alpha}$  and ignoring the irrep index  $l$ . The measure  $d\mu(\vec{\xi})$  is, for each case,

$$\begin{aligned} d\mu(\vec{\gamma}) &= \prod_n \frac{2j+1}{\pi} \frac{d^2\vec{\gamma}_n}{(1+|\gamma_n|^2)^2} \\ d\mu_P(\vec{\eta}) &= \prod_n \frac{2k-1}{\pi} \frac{d^2\vec{\eta}_n}{(1-|\eta_n|^2)^2} \\ d\mu_{BG}(\vec{\eta}) &= \prod_n \frac{2}{\pi} I_{2k-1}^2(2|\eta_n|) d^2\vec{\eta}_n \\ d\mu(\vec{\alpha}) &= \prod_n \frac{d^2\vec{\alpha}}{\pi} \end{aligned} \tag{A.2}$$

for  $SU(2)$  coherent states, Perelomov  $SU(1, 1)$  coherent states and Barut–Girardello coherent states, and harmonic oscillator coherent states, respectively.

If we choose  $l = j$  and

$$f(\vec{\xi}) = \frac{1}{\sqrt{2}} \left[ \exp\left(-i\frac{\pi}{4}\right) \delta(\vec{\xi} + i\vec{\gamma}) + \exp\left(i\frac{\pi}{4}\right) \delta(\vec{\xi} - i\vec{\gamma}) \right] \tag{A.3}$$

the general state (A.1) will reduce to the entangled  $SU(2)$  coherent state (22). All examples of coherent states considered in this paper can be constructed accordingly.

As an interesting example, we consider the multiple qubit entanglement. Shor [1] introduced the quantum Fourier transforms in order to apply quantum computation to factorize a number  $a$ , with  $0 \leq a \leq q$ . This number  $a$  can be expressed in qubits as

$$|a\rangle \equiv |\vec{\varepsilon}\rangle = |\varepsilon_1, \varepsilon_2, \dots, \varepsilon_N\rangle \quad \varepsilon_i \in \{0, 1\} \tag{A.4}$$



where  $\bar{\varepsilon}$  is the binary notation of  $a$ . Here  $N$  is the lowest integer greater than or equal to  $\log_2 q$ . The state  $|a\rangle$  is a product multiparticle  $SU(2)$  coherent state for  $j = 1/2$ . The quantum Fourier transform of the state  $|a\rangle$  is

$$q^{-1/2} \sum_{c=0}^{q-1} \exp(i2\pi ac/q) |c\rangle. \quad (\text{A.5})$$

The state  $|c\rangle$  is also a product state for all  $c$  such that  $0 \leq c \leq q - 1$ . We can express the transformed state as the entangled  $SU(2)$  coherent state (A.1) such that

$$q^{-1/2} \sum_{\bar{\varepsilon}_i \in \{0,1\}} \exp(i2\pi ac/q) |\bar{\varepsilon}\rangle = q^{-1/2} \sum_{\vec{\gamma}_i \in \{0,\infty\}} \exp(i2\pi ac/q) |j\vec{\gamma}\rangle \quad (\text{A.6})$$

where  $c \equiv \bar{\varepsilon}$ . The above state can also be written in the form of (A.1) with

$$f(\vec{\xi}) = q^{-1/2} \sum_{\vec{\gamma}_i \in \{0,\infty\}} \exp(i2\pi ac/q) \delta(\vec{\xi} - \vec{\gamma}_i). \quad (\text{A.7})$$

Hence the Fourier transform state (A.5) can be treated within the formalism of entangled  $SU(2)$  coherent states.

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